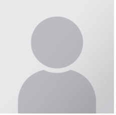
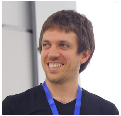
Linear Equations with Min and Max Operators: Computational Complexity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Optimization problem

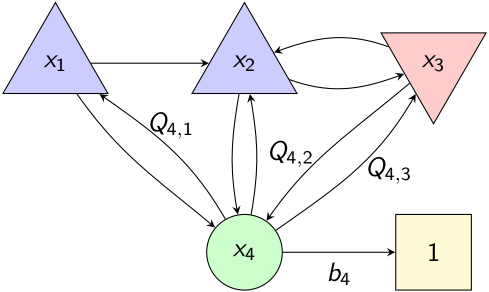
Linear equations with Min and Max operators (LEMM)
Consider the following optimization problem on $\mathbb{R}^n$

$$\begin{cases} \min_{\boldsymbol{x} \in \mathbb{R}^n} & \boldsymbol{c}^\top \boldsymbol{x} \\ s.t. & x_i = \min\{x_\ell : \ell \in N(i)\} & i \in [n_1] \\ & x_i = \max\{x_\ell : \ell \in N(i)\} & i \in [n_2] \setminus [n_1] \\ & x_i = Q_{i,:}\boldsymbol{x} + b_i & i \in [n] \setminus [n2 + n_1] \end{cases}$$

where $N \colon [n] \to 2^{[n]}$ defines the neighbors of a variable and there is one constrain per variable.

Generality: LEMMs can model
finite nested linear, min, and max operators and
boolean variables.

Linear, min, max restrictions as a graph



An alternative representation of the restrictions is given by a graph.

Vertices represent variables.

They are partitioned into min, max, or linear types.

Edges of linear vertices have weights.

An extra node (square) describes the constants $\boldsymbol{b}$.

LEMM decision problem

Instead of solving the optimization problem and computing its value and an optimal solution, we consider the following decision problem.

Definition (LEMM decision problem)

Given an LEMM and a threshold $t \in \mathbb{R}$, decide whether

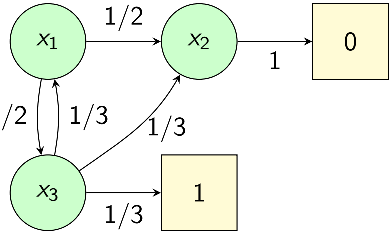
$$\mathrm{val}(LEMM) < t\,.$$

Examples

Markov Chains as LEMMs

In a Markov Chain with a target state,
the value is the probability of eventually reaching the target.
There is a linear reduction to express the value of a Markov Chain as the unique solution of a system of linear equations.

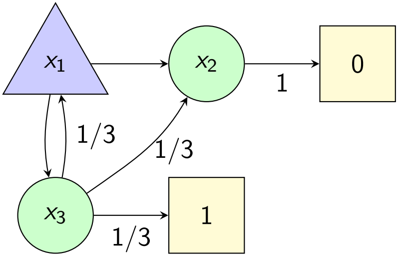
In a formulation as an LEMM
the linear constraints are a convex combination of
$\{x_1, x_2, \ldots, x_n, 0, 1\}$.



Markov Decision Processes as LEMMs

In a Markov Decision Process (MDP) with a target state, the value is the probability of eventually reaching the target. After a linear preprocessing, the value of an MDP is the value of a linear program.

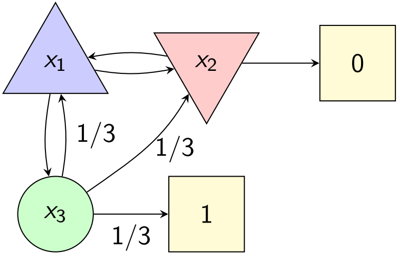
A formulation as an LEMM uses the max operator to express everything as equalities and the linear constraints are convex combinations.



Simple Stochastic Games as LEMMs

In a Simple Stochastic Game (SSG),
the value is the probability of eventually reaching the target.

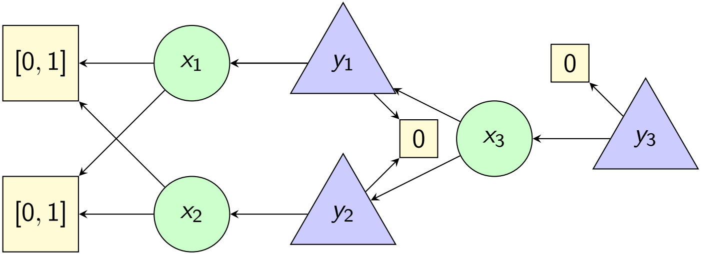
A formulation as an LEMM uses
the min and max operators and
the linear constraints are convex combinations.



Neural network verification

For a multilayer neural network with ReLU or Maxout activations, decide whether the output is below a threshold for all inputs in a polygon.

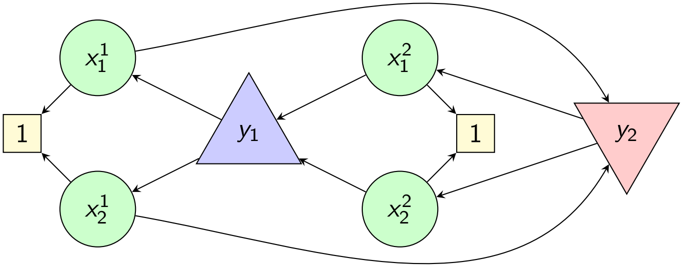
A formulation as an LEMM uses min, max, and linear operators to define the polygon, the max operators for activation functions and the linear constraints for the weights.



Capital preservation

Periodic investments where first the controller chooses a strategy and then the assets evolve in the market .
The minimum operator is the worst case scenario for the market.

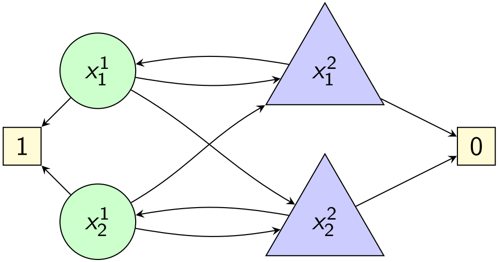
A formulation as an LEMM uses
min and max for the market and the controller
linear operators model assets' evolution, so they are positive.
If repeating the cycle converges, then capital is preserved over time.



Co-evolution in ecosystem

Species evolve according to internal interactions and external interventions.
Evolution is approximated by linear interactions.
Under no external interventions, all species go extinct.

A formulation as an LEMM uses
the max operator to represent population
linear operators admit positive and negative interactions.
The solution of the LEMM represents the stable population.



Results

Conditions on constraints

C1 **Halting or Stability**
Every convex combination of underlying linear equations defines a matrix $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$, and they all satisfy

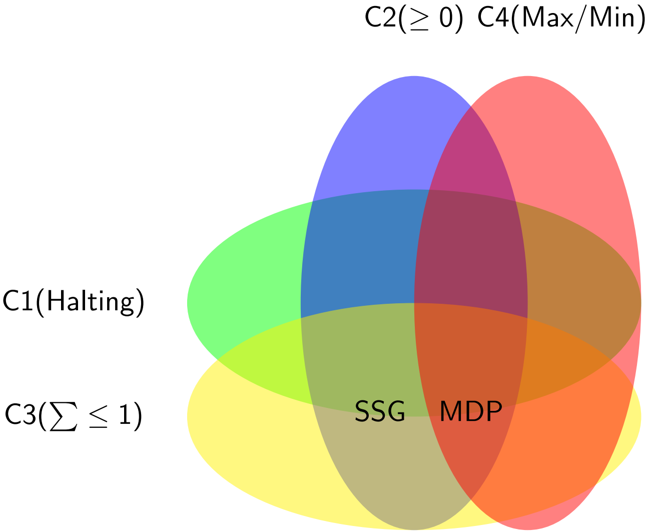
$$\lim_{m \to \infty} \boldsymbol{Q}^m = \mathbf{0}_{n \times n} \,.$$

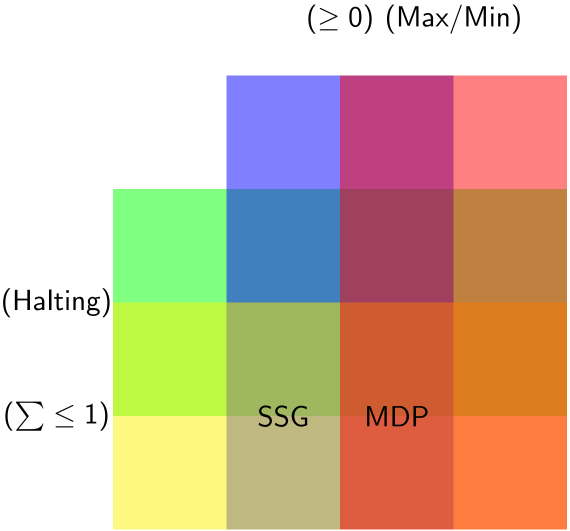
C2 **Non-negative coefficients**
For all $i \in [n] \setminus [n_1 + n_2]$, we have $Q_{i,:} \geq 0$ and $b_i \geq 0$.

C3 **Sum up to 1**
For all $i \in [n] \setminus [n_1 + n_2]$, we have $Q_{i,:}\mathbf{1} + b_i \leq 1$.

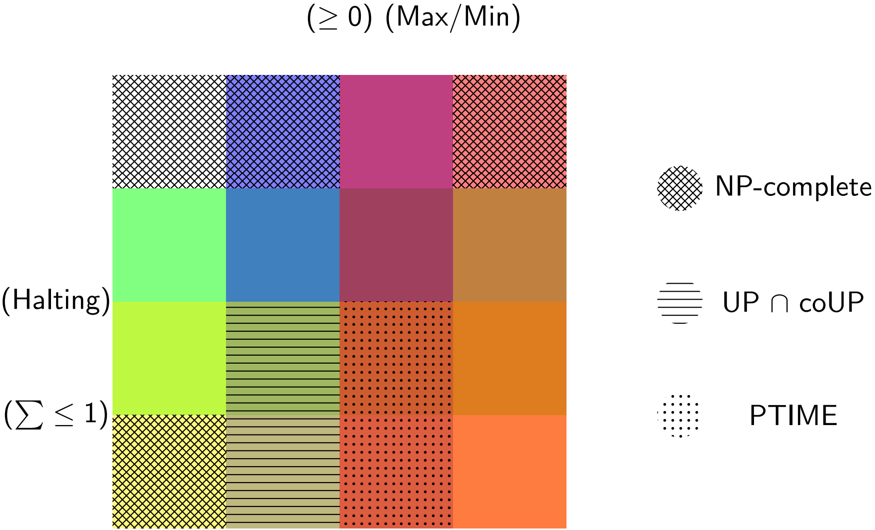
C4 **Max-only or min-only**
$n_1 = 0$ or $n_2 = 0$.

Conditions and models

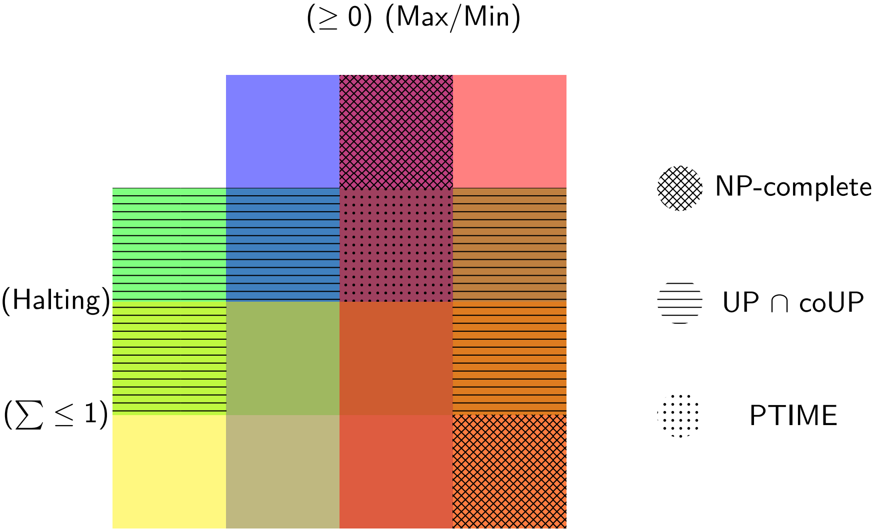



(≥ 0) (Max/Min)
(Halting)
(∑ ≤ 1)
SSG
MDP

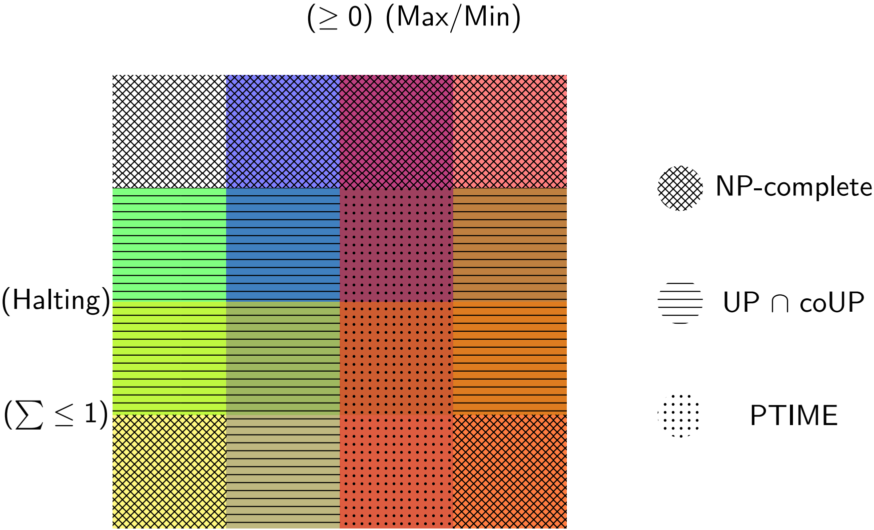
Previous results



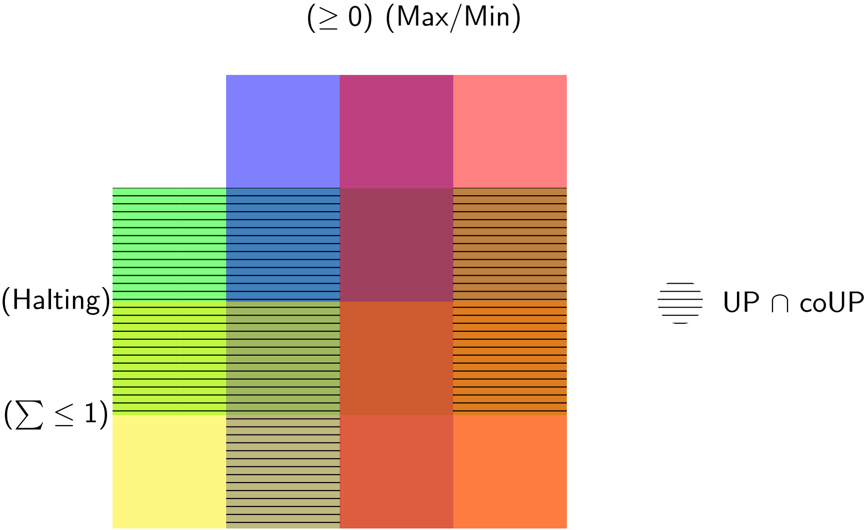
Our results: Complexity



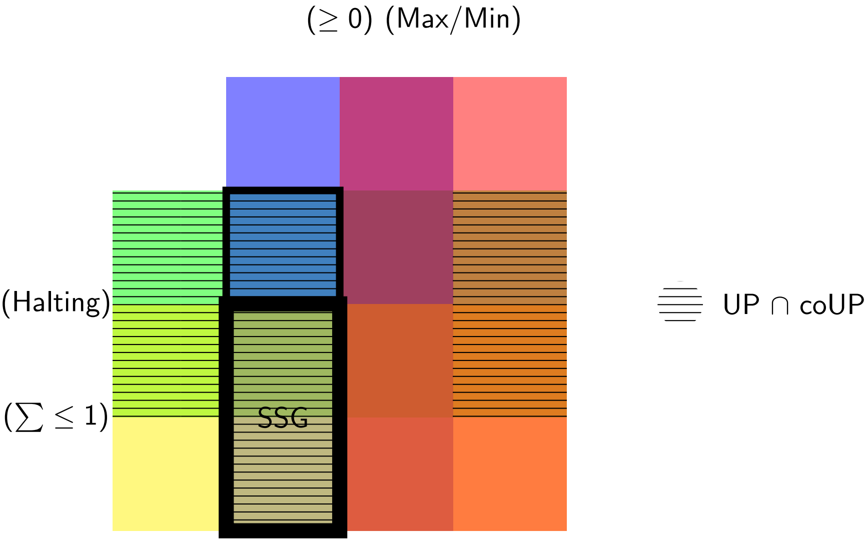
Complete characterization



Our results: Polynomial equivalent classes



Our results: Polynomial equivalent classes

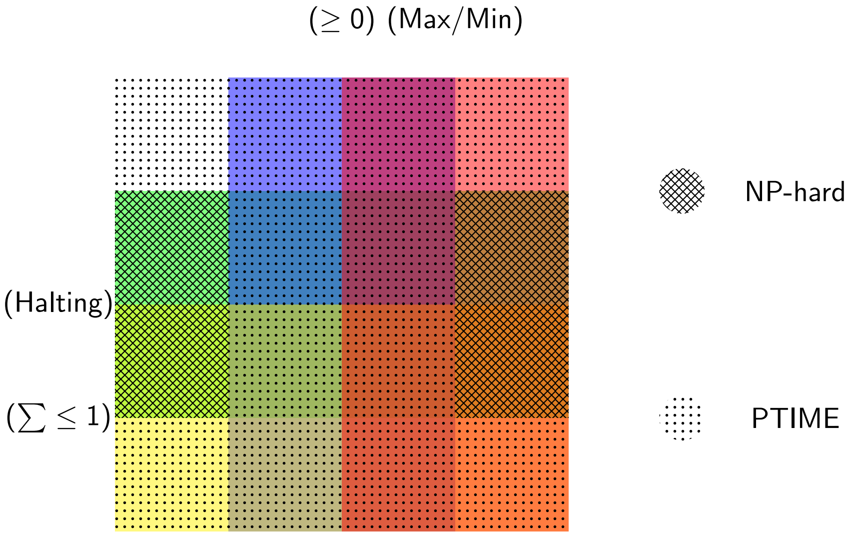


Checking conditions

Definition (Condition decision problem)

Given an LEMM and a subset of conditions,
determine whether all the conditions are satisfied.

Checking conditions: Complexity



Key technical result

Unique solution

Theorem

Every LEMM satisfying condition C1 (Halting) has a unique solution.

Thank you!